On the Convergence of Blind Over-Sampled Decision Feedback Equalizers

Leszek L. Szczecinski, Ado Gei
Departamento de Ingeniería Eléctrica
Universidad de Chile
{leszek,agei}@cec.uchile.cl

Abstract: *Decision Feedback Equalizers (DFEs) efficiently reduce the effect of Inter Symbol Interference (ISI). Their blind (or Decision-Directed) adaptation may converge to so-called degenerated solutions when signal at the equalizer output is independent of its input. One of the solution to this problem is to use Predictive Decision Feedback Equalizer (PDFE), another solution proposed consists in imposing constrains on the coefficients of the feed-forward filter. These solutions are unsatisfactory so a new simple algorithm of blind adaptation is proposed. It consist in constraining the feedback coefficients and guarantees the convergence to desired solution. The convergence properties are shown theoretically and simulations results are presented.*

Keywords: constant modulus algorithm, blind equalization, decision feedback

INTRODUCTION

Channel equalization is one of fundamental problems in digital telecommunications. It consists in removing the *Intersymbol Interference* (ISI) from data received through a telecommunications channel. The mathematical SIMO linear model of telecommunication system including time over-sampling (fractionally sampled receivers [5]) and space over-sampling (multiple antennas receivers [7]) is shown in Fig. 1a.

The signal received in each sub-channel is modeled as:

$$\tilde{r}_{(k)}(n) = H_{(k)}(q^{-1})s(n) + e_{(k)}(n) \qquad (1)$$

where the FIR linear filter modeling each of the sub-channels are expressed as polynomials:

$$H_{(k)}(q^{-1}) = \sum_{m=0}^{M_H-1} h_{(k),m} q^{-m} \qquad (2)$$

of a unit delay operator q^{-1}.

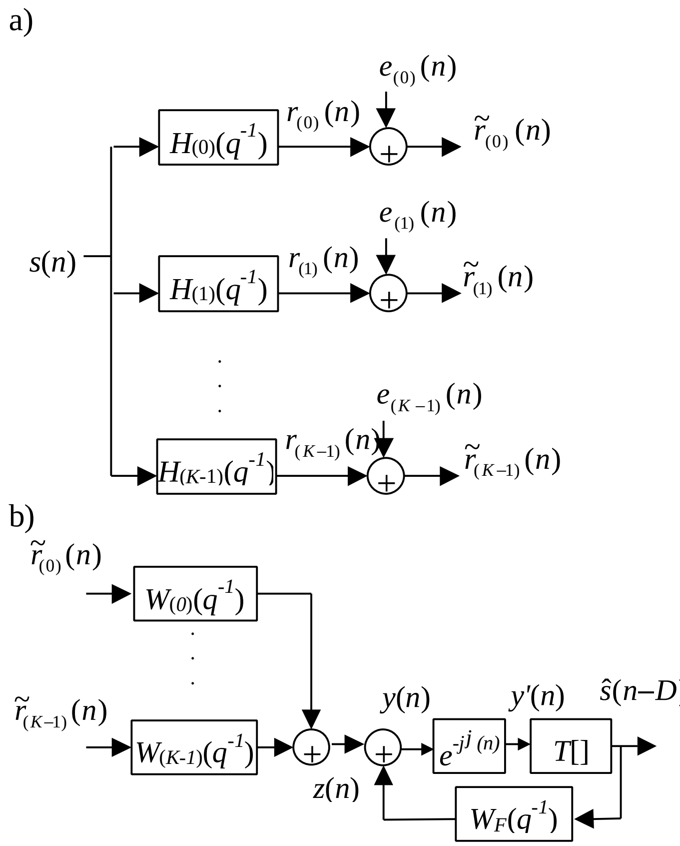

Fig. 1 Model of a) SIMO telecommunication channel with *K* sub-channels, b) MISO Decision Feedback Equalizers with *K* feed-forward filters.

The signal sent $s(n)$ is modeled as white noise taking values from the set S. In this study we limit simulations and examples to 4QAM modulation:

$$S = \{-1, 1, j, -j\} \qquad (3)$$

Random errors $e_{(k)}(n)$ corrupting signals $r_{(k)}(n)$ are modeled as white, uncorrelated noises.

This research was supported by Comisión Nacional de Investigación Científica y Tecnológica CONICYT, Chile (FONDECYT Projects Nº1990597, Nº1000903). Please address correspondence to: Leszek Szczecinski, Email: Leszek @cec.uchile.cl

The model of Decision Feedback Equalizer is shown in Fig. 1b. The coefficients of a feedback and K feed-forward filters :

$$W_F\left(q^{-1}\right)=w_{F,1}q^{-1}+\ldots+w_{F,M_F}q^{-M_F}$$

$$W_{(k)}\left(q^{-1}\right)=w_{(k),0}+w_{(k),1}q^{-1}+\ldots+w_{(k),M-1}q^{-M+1}.$$

are gathered in vectors $\boldsymbol{w}_{FB}$ and $\boldsymbol{w}_{FF}$.

The symbol received is estimated using the filtered signal $y'(n)$ and the decision device $T[]$:

$$\hat{s}(n)=T\left[y'(n)\right]=\arg_s min\left\{\left\|y'(n)-s\right\|, s\in\Sigma\right\} \qquad (4)$$

where $y'(n)=y(n)e^{j\varphi(n)}$. Multiplication by a constant $e^{-j\varphi(n)}$ – so called phase tracking – is necessary in blind adaptation. It is done using standard algorithm [8; Ch. 10]:

$$\varphi(n+1)=\varphi(n)+\mu_\varphi \operatorname{Im}\left(y(n)\hat{s}^*(n-D)\right) \qquad (5)$$

If equalizers are functioning in supervised mode, their coefficients are found using known training sequences $s(n)$ through minimization of some criterion. The most popular one is Least Mean Squares (LMS) [9] applied to the difference between the equalizer output $y(n)$ and the desired signal $s(n-D)$ with the delay D chosen to optimize the equalizer's performance (its adequate choice is a problem not solved in literature and will not be addressed here). In DFE, $\boldsymbol{w}_{FF}$ and $\boldsymbol{w}_{FB}$ are optimized *jointly* using LMS.

BLIND ADAPTATION

In case of time-varying channels, equalizer requires periodic training to follow the changing environment; the drawback lies in fact that training signals sent reduce effective channel bandwidth. Blind adaptation, in contrast, allows the equalizer to estimate its coefficients without knowledge of the transmitted sequences. The most popular ones are based on minimization of *Constant Modulus* (CM) criterion applied to the signal $y(n)$ [4]

$$\hat{\boldsymbol{w}}=\arg_w \inf\left\{E\left[\left(\left|y(n)\right|^2-1\right)^2\right]\right\} \qquad (6)$$

which in case of stochastic gradient optimization results in *Constant Modulus Algorithm* (CMA). Blind adaptation of DFE is an extension of supervised DFE in the sense that only cost function is being changed: feed-forward and feedback coefficients are *jointly* adapted using CM cost functions [4]. The algorithm, called herein DFE_CMA, is defined as follows:

$$\begin{aligned} &y(n)=\boldsymbol{w}^H(n)\boldsymbol{u}(n)\\ &e(n)=y(n)\left(1-\left|y(n)\right|^2\right)\\ &\boldsymbol{w}(n+1)=\boldsymbol{w}(n)+\mu\boldsymbol{u}(n)e^*(n)\end{aligned} \qquad (7)$$

where

$$\begin{aligned} &\boldsymbol{u}(n)=\begin{bmatrix}\tilde{\boldsymbol{r}}_{(0)}(n)\\ \vdots\\ \tilde{\boldsymbol{r}}_{(K-1)}(n)\\ \hat{\boldsymbol{s}}_F(n)\end{bmatrix}, \boldsymbol{w}(n)=\begin{bmatrix}\boldsymbol{w}_{FF}(n)\\ \boldsymbol{w}_{FB}(n)\end{bmatrix}\\ &\tilde{\boldsymbol{r}}_{(k)}(n)=\left[\tilde{r}_{(k)}(n),\ldots,\tilde{r}_{(k)}(n-M+1)\right]^T\\ &\hat{\boldsymbol{s}}_F(n)=\left[\hat{s}(n-D-1),\ldots,\hat{s}(n-D-M_F)\right]^T\end{aligned} \qquad (8)$$

Some solutions obtained by means of the algorithms DFE_CMA are not valid for the equalization purpose [2]; they will be called herein *degenerated solutions*. For example, such solution may take the form [6]:

$$\boldsymbol{w}_{FF}(n)=\boldsymbol{0}, \quad \boldsymbol{w}_{FB}(n)=\left[0,\ldots0,1,0,\ldots,0\right]^T \qquad (9)$$

For this set of coefficients $y(n)$ is independent of $\tilde{r}_{(k)}(n)$; also it belongs to Σ $(\left|y(n)\right|^2=1)$ and as a consequence weights $\boldsymbol{w}$ are not being modified (cf. Eq.(7)). The same problem may appear if equalizer is operating in *Decision Directed* (DD) mode [9].

In order to avoid convergence to degenerated solutions, two approaches have been proposed in the literature. One involves reformulation of the feedback equalizer in the form of Predictive DFE (PDFE) adapted blindly [10] using CM cost function. This algorithm yields results inferior to those obtained by means of the algorithm DFE (although theoretically they are equivalent).

Second algorithm (or rather group of algorithms) imposes constraints on the equalizer's coefficients in order to avoid degenerated solutions. An algorithm representative for the class of algorithms proposed, called herein DFE_CMA_FF, imposes the constraints on the power of feed-forward coefficients [6] :

$$\hat{\boldsymbol{w}}=\arg_{\boldsymbol{w}}\inf\left\{E\left[\left(\left|y(n)\right|^2-1\right)^2\right]\ \left|\ \left\|\boldsymbol{w}_{FF}\right\|^2>q\right.\right\} \qquad (10)$$

It is shown (Appendix) that this algorithm may result in degenerated solutions in case of over-sampled channels, independently of chosen parameter q. This means that any “soft” constraint imposed on the feed-forward coefficient is not satisfactory; “soft” constraints are all which do not fix the values of the coefficients.

In this paper, new algorithm is proposed, called herein DFE_CMA_FB. It imposes the constraints on the feedback coefficients and guarantees convergence to correct solutions; it is defined through the following problem of constrained minimization:

$$\hat{\boldsymbol{w}}=\arg_{\boldsymbol{w}}\inf\left\{E\left[\left(\left|y(n)\right|^2-1\right)^2\right]\ \left|\ \left\|\boldsymbol{w}_{FB}\right\|^2<E\left[\left|z(n)\right|^2\right]\right.\right\} \qquad (11)$$

If solution of Eq.(11) is found using stochastic gradient optimization it results in the following algorithm:

$$\begin{aligned}
&y(n)=\boldsymbol{w}^H(n)\hat{\boldsymbol{s}}_F(n)\\
&e(n)=y(n)\left(R-\left|y(n)\right|^2\right)\\
&E_Z(n)=\alpha_Z E_Z(n-1)+(1-\alpha_Z)\left|z(n)\right|^2\\
&\boldsymbol{w}_{FF}(n+1)=\boldsymbol{w}_{FF}(n)+\mu\tilde{\boldsymbol{r}}(n)e^*(n)\\
&\boldsymbol{w}_{FB}(n+1)=\boldsymbol{w}_{FB}(n)\left[1-\lambda\mu\right]+\mu\boldsymbol{s}_F(n)e^*(n)
\end{aligned} \qquad (12)$$

where λ is the Lagrange multiplier: $\lambda = \lambda_0 > 0$ if $E_Z(n)\le\left\|\boldsymbol{w}_{FB}(n)\right\|^2$ ($\lambda_0 = 2$ in this study) and $\lambda = 0$ if $E_Z(n)>\left\|\boldsymbol{w}_{FB}(n)\right\|^2$. A forgetting factor α_l (α_l = 0.95 in this study) is used to estimate the value of $E\left[\left|z(n)\right|^2\right]$.

To explain the origin of the algorithm DFE_CMA_FB, $z(n)$ is presented as a convolution of the channel and feed-forward filters with the input signal $s(n)$:

$$\begin{aligned}
z(n)&=s(n)\sum_{k=0}^{K-1}H_{(k)}\left(q^{-1}\right)W^*_{(k)}\left(q^{-1}\right)=s(n)C\left(q^{-1}\right)=\\
&=\sum_{k=0}^{D-1}c_k s(n-k)+c_D s(n-D)+\sum_{k=1}^{M_C-D-1}c_{k+D}s(n-D-k)
\end{aligned} \qquad (13)$$

where $M_C=M+M_H-1$ is the order of $C(q^{-1})$ and D is unknown (because of blind adaptation) estimation delay. Decision feedback improves the solution since it allows to eliminate the last term in Eq.(13) if :

$$w^*_{F,k}=-c_{k+D},\quad for\quad k=1,\ldots,M_F \qquad (14)$$

This implies that the following condition has to be satisfied :

$$\left\|\boldsymbol{w}_{FB}\right\|^2=\sum_{k=1}^{M_F}\left|w_{F,k}\right|^2<\sum_{k=0}^{M_C-1}\left|c_k\right|^2 \qquad (15)$$

The second term of the inequality Eq.(15) may be expressed using variance of $z(n)$, due to assumption of signal $s(n)$ being white and having unit power ($E\left[\left|s(n)\right|^2\right]=1$):

$$\sum_{k=0}^{M_C-1}\left|c_k\right|^2+\left\|\boldsymbol{w}_{FF}\right\|^2\sigma_e^2=E\left[\left|z(n)\right|^2\right] \qquad (16)$$

Thus in general the following condition should be satisfied:

$$\left\|\boldsymbol{w}_{FB}\right\|^2<E\left[\left|z(n)\right|^2\right]-\left\|\boldsymbol{w}_{FF}\right\|^2\sigma_e^2 \qquad (17)$$

The effect of noise was not considered in the development of the algorithm defined in Eq.(11) and should be included if value of σ_e^2 is not negligible.

SIMULATIONS

Channel used in simulations was defined through the position of roots of polynomials $H_{(k)}\left(q^{-1}\right)$ as in [3]; the roots for each sub-channel are shown in Table 1. The coefficients in Eq.(2) were normalized :

$$\sum_{m=0}^{M_H-1}\left|h_{(k),m}\right|^2=1 \qquad (18)$$

Note that sub-channels’ roots are close to each other and close to the unit circle which makes equalization a difficult task, requiring a decision feedback [5][6].

Table 1. Roots of the polynomials representing linear sub-channels in the model from Fig. 1a.

sub-channel	roots
0	$0.95e^{(j\pi 0.60)}$, $1.0e^{(j\pi 0.95)}$, $1.0e^{(-j\pi 0.55)}$
1	$0.95e^{(-j\pi 0.30)}$,$1.0e^{(j\pi 1.00)}$,$1.0e^{(-j\pi 0.50)}$

The signal $s(n)$ (4QAM) was simulated using pseudo-random numbers belong to the set S. Noises $e_{(k)}(n)$ k=0,1 were simulated using pseudo-random normally-distributed numbers with variance s_e^2 which defined the *Signal-To-Noise-Ratio* (SNR) at the output of each of the sub-channels:

$$SNR = -20\log_{10} s_e \qquad (19)$$

In order to show properties of the studied algorithms the degenerated solutions were searched for using the algorithm DFE_CMA. Next the algorithm DFE_CMA_FF was applied with the constant q which resulted to degenerated solution. It is sufficient to show *one* case, to show that solutions obtained constraining the feed-forward coefficients *may* be degenerated (which of course also depends on the optimization method chosen).

Parameters of the equalizer (M and M_F) as well as the adaptation steps were chosen through an extensive search; the following values were found to yield the optimal results: $M = 6$, $M_F = 3$, $m = 0.001$, $m_j = 0.01$.

A degenerated case is analyzed in Fig. 2 for each of three algorithms: DFE_CMA, DFE_CMA_FF (with parameter $q = 1$) and DFE_CMA_FB. One may observe that $E_Z(n)$ tends to zero (i.e. the solution becomes degenerated) if the algorithms DFE_CMA or DFE_CMA_FF are used (the latter despite of constraints applied). Note that the received signal may seem correct in case of the algorithm DFE_CMA_FF (cf. Fig. 2d) but in fact is not related to signal $s(n)$ ($SER = 0.75$). On the other hand, the proposed algorithm DFE_CMA_FB results in a correct solution ($SER<10^{-4}$).

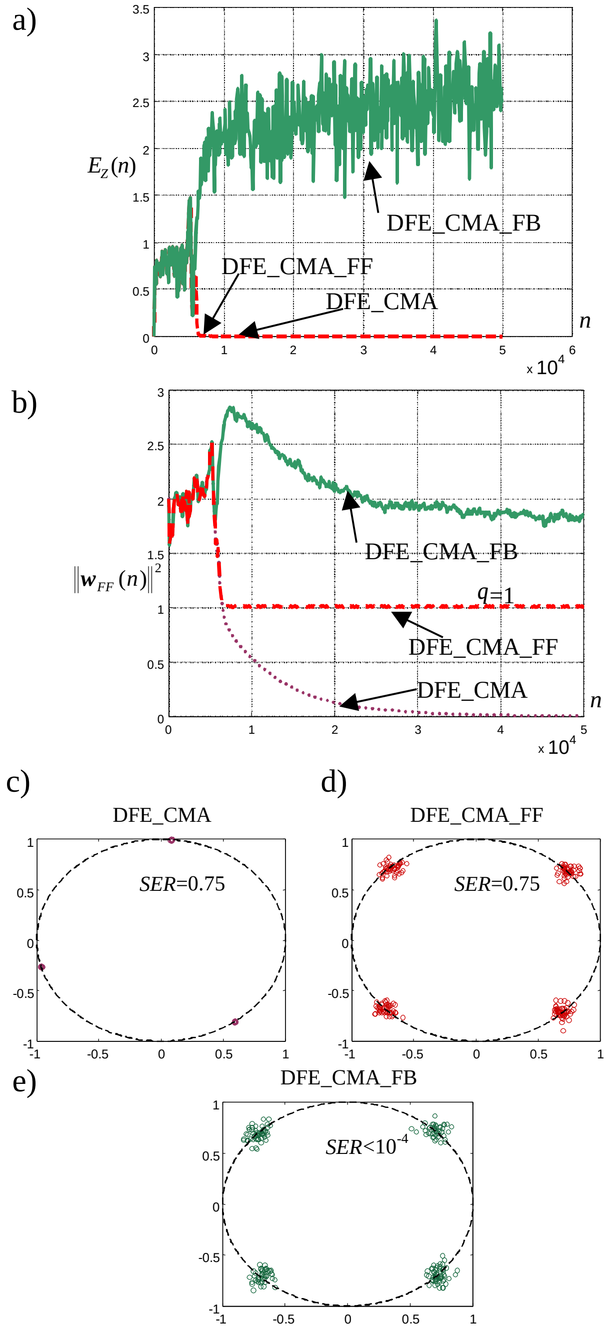

Fig. 2 Results obtained by means of studied algorithms applied to a signal resulting in degenerated solution (in CMA_DFE case) for *SNR*=25dB, a) variation of $E_Z(n)$, b) variation of $\|\mathbf{w}_{FF}(n)\|^2$, c-e) $y'(n)$ obtained by means of the algorithm DFE_CMA, DFE_CMA_FF and DFE_CMA_FB.

CONCLUSIONS

In this paper blind adaptation of Fractionally Spaced Decision Feedback Equalizer using

Constant Modulus Algorithm was analyzed. The problem of convergence to so-called degenerated solution was addressed. The degenerated solutions are defined as such set of coefficients for which equalizer's output is independent of its input.

The existing algorithms which should solve the problem are

- Predictive Decision Feedback Equalizer with CMA adaptation (called PDFE_CMA) which changes form of equalizer and adapts separately feed-forward and feedback coefficients.
- The DFE_CMA algorithm which imposes constraints on coefficients of the feed-forward filters (called here DFE_CMA_FF).

These algorithms result in unsatisfactory results : or fail to guarantee the convergence (DFE_CMA_FF) or are sub-optimal (PDFE_CMA).

New algorithm DFE_CMA_FB is proposed. which guarantees convergence to the correct (not degenerated) solution. It's essential part are constraints imposed on the feedback filter. Thus the convergence problem was solved in this paper through a simple algorithm which may be applied in blind (for example Constant Modulus Algorithm) or Decision Directed adaptation.

Appendix

It will be shown that constraining of feed-forward coefficients may result in *degenerated solutions*. Also, it will be demonstrated that solution obtained by means of the new algorithm DFE_CMA_FB may not be degenerated.

The *degenerated solution* is defined as such set of coefficients $\boldsymbol{w}_{FF}$ and $\boldsymbol{w}_{FB}$ that $y(n)$ is not affected by the received signals $\tilde{r}_{(k)}(n)$.

First the noiseless case (i.e. $s_e = 0$) is considered. The solution is degenerated if $z(n) \equiv 0$ i.e. for combined channel – feed-forward response $C(q^{-1}) \equiv 0$ (cf. Eq.(13)). This condition may be reformulated using vector notation :

$$\boldsymbol{c}=\sum_{k=0}^{K-1}\boldsymbol{H}_{(k)}\boldsymbol{w}^{*}_{(k)}=\boldsymbol{0} \qquad (20)$$

were $\boldsymbol{H}_{(k)} \in C^{M_C x M}$ for $k = 0,...,K$-1 are convolution matrix [5]. Using notation $\boldsymbol{H} = \left[\boldsymbol{H}_{(0)},\ldots,\boldsymbol{H}_{(K-1)}\right] \in C^{M_C x KM}$, the degenerated solution occurs when

$$\boldsymbol{c}=\boldsymbol{H}\boldsymbol{w}^{*}_{FF}=\boldsymbol{0} \qquad (21)$$

The above equation has a trivial solution $\boldsymbol{w}_{FF} = \boldsymbol{0}$ but also non-trivial ones if

$$M_C = M_H+M-1 < KM \qquad (22)$$

condition Eq.(22) may be satisfied in over-sampled channels (K>1). These degenerated solutions may have arbitrary norm $\|\boldsymbol{w}_{FF}\|$ so constraints used in Eq.(10) will be satisfied for any value of q. ■

Now, if noise is added, the output $y(n)$ is composed of two terms:

$$y(n) = \boldsymbol{w}^{H}_{FB}\hat{\boldsymbol{s}}_F(n) + v(n) \qquad (23)$$

where $v(n)$ is the colored noise with variance $s_v^2 = \|\boldsymbol{w}_{FF}\|^2 s_e^2 = q s_e^2$. The algorithm CMA will tend to eliminate the influence of errors $v(n)$ (so that normalized kurtosis of the output $y(n)$ decreases [5]). This may be obtained if $\|\boldsymbol{w}_{FF}\|$ is minimized, but since $\|\boldsymbol{w}_{FF}\|^2$ is constrained to q, the solutions obtained will not change (will be degenerated). ■

On the other hand, if condition from Eq.(17) is satisfied it is obviously not possible to obtain a degenerated solution which will be demonstrated by contradiction (for noiseless case). Suppose that a solution is degenerated. Then $E\left[|z(n)|^2\right] = \|\boldsymbol{w}_{FF}\|^2 s_e^2$ and - due to constraints imposed in Eq.(17) - $\|\boldsymbol{w}_{FB}\| = 0$. Since all coefficients of feedback filter are zero, $y(n)$ is independent of the output $\hat{s}(n)$ so no feedback exist; since only feedback may cause degenerated solutions, the solution obtained may not be degenerated. ■

REFERENCES

[1] C. Belfiore, J. H. Park Jr. "Decision Feedback Equalization", Proc. IEEE Vol. 67, No. 8, August 1979, pp. 1143-1156.

[2] R. A. Casas, C. R. Johnson Jr., R. A. Kennedy, Z. Ding, R. Malamut "Blind Adaptive Decision Feedback Equalization: a Class of Channels Resulting in Ill-convergence From a Zero Initialization", Intl. J. Adaptive Control and Signal Processing, Vol. 12, 1998, pp. 173-193.

[3] T. J. Enders, S. D. Harford, C. R. Johnson, G. B. Giannakis "Blind Adaptive Channel Equalization Using Fractionally-Spaced Receivers: A Comparison Study", Proc. Conference on Information Sciences and Systems, Princeton N.J., USA, March 1996.

[4] D. N. Godard "Self-Recovering Equalization and Carrier Tracking in Two-Dimensional Data Communication Systems", IEEE Trans. Comm., Vol. COM-28, No. 11, Nov. 1980, pp. 1867-1875.

[5] C. R. Johnson Jr., P. Schniter, T. J. Enders, J. D. Behm, D. R. Brown, R. A. Casas "Blind Equalization Using the Constant Modulus Criterion: A Review", Proc. IEEE, Vol. 86, No. 10, Oct. 1998, pp. 1927-1950.

[6] C. B. Papadias and A. Paulraj "Decision-Feedback Equalization and Identification of Linear Channels Using Blind Algorithms of the Bussgang Type" 29th Annual Asilomar Conference on Signals, Systems and Computers, Oct. 30-Nov. 1, 1995, Pacific Grove, California, USA.

[7] A. Paulraj, C. B. Papadias, "Space-Time Processing for Wireless Communications", IEEE Signal Process. Mag., Nov. 1997, pp. 49-83.

[8] J. Proakis, Digital Communications, McGraw-Hill, 1995.

[9] S. U. H. Quereshi "Adaptive Equalization", Proc. IEEE, Vol. 73, No. 9, Sept. 1985, pp. 1349-1385.

[10] L. Tong, D. Liu "Blind Predictive Decision-Feedback Equalization via the Constant Modulus Algorithm", Proc. ICASSP'97, Munich, Germany, April 1997, pp. 3901-3904.