IMPOSED WINDOW ESTIMATE FOR ADAPTIVE EXTRAPOLATION OF COMPLEX SINUSOIDS IN GAUSSIAN NOISE

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ABSTRACT

The Adaptive Weighted Norm Extrapolation (AWNE) method is well suited for recovering sinusoidal signals from short data records. The time-domain shape of the envelope of the result is considered to be due to an imposed windowing on the original signal. In this paper, we estimate the relevant imposed windows when the input is a sum of sinusoids corrupted by additive Gaussian noise. We assume that the AWNE method produces two independent contributions from the signal and from the noise to derive the form of the imposed window which is verified experimentally. This paper also establishes the role of a squared Tukey window in the imposed window. Finally, the use of the window estimate in SAR profile extrapolation is described and illustrated.

1. INTRODUCTION

The AWNE method [1] produces a signal that extends beyond the limited range of the given data by inferring the frequency contents from this finite-record of data samples. AWNE preserves the given data and extends it at the two ends. This algorithm is closely related to re-weighting methods of signal reconstruction with sparseness constraints [2] and has been shown to be well suited to produce superresolution in Synthetic Aperture Radar (SAR) image formation [3]. Besides the high resolution characteristic, an interesting behavior of AWNE is the slow decay to zero of the tails of the extrapolated signal. This behavior is similar to the application of a window to a signal.

In [4], we characterized this decay to zero as a windowing effect on the original signal for a sum of sinusoids and found that the squared Tukey window modeled this effect very closely. In this paper, we extend the study of this windowing effect to the case where the sinusoidal input signal is corrupted with additive noise. The interest in the noisy case is to extend the analysis to SAR phase history profile extrapolation. The AWNE imposed window model can be used to have better control of the sidelobe behavior in SAR image formation.

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2. THE AWNE METHOD

The AWNE method extrapolates a signal through an iterative process of frequency norm modification and minimum norm extrapolation [1]. Per iteration, the AWNE method estimates the signal's autocorrelation and then creates a linear expansion in terms of shifted copies of this estimate and which matches the given data samples in the given range. This linear expansion involves a set of extrapolation coefficients that are calculated by solving a set of linear equations.

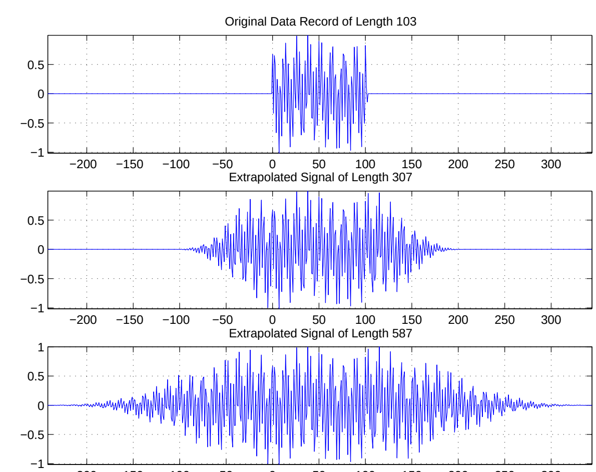

Figure 1: Examples of AWNE extrapolation of complex sinusoid (real part shown).

Going from iteration k to $k+1$, the key equations involved are:

$$\begin{aligned} h_k[n] &= x_k[n] \cdot p[n] &(1)\\ q_{k+1}[n] &= h_k^*[-n] * h_k[n] &(2)\\ x_{k+1}[n] &= \sum_{i=0}^{L-1} b_i \cdot q_{k+1}[n-i] &(3)\end{aligned}$$

where $x_k[n]$ is the current extrapolated signal, $p[n]$ is the algorithm window used in the estimation of the autocorrelation $q_{k+1}[n]$. The b_i's are the extrapolation coefficients which are solved for by enforcing $x_{k+1}[n] = x[n]$ for $n = 0, 1, 2, ..., L-1$ in Eqn.(3). In most cases of interest, the AWNE algorithm converges to a fixed result $x_e[n]$ after a small number of iterations. The length of $x_e[n]$ is $N_e = 2J + L - 2$ where J is the length of $p[n]$. A slow decay to zero of the result can be predicted from Eqn.(3), due to the decaying shape of autocorrelations computed from finite duration sequences. Figure 1 shows examples of AWNE extrapolations for a test signal consisting of a sum of 3 complex sinusoids. Figure 1 (top) is the original data record of length $L = 103$, the middle plot is the extrapolated output of length $N_e = 307$ and the bottom plot shows the result for $N_e = 587$. Notice that the extrapolated signals behave like windowed versions of the original signal.

3. IMPOSED WINDOW ESTIMATION WITHOUT NOISE

The AWNE method is well suited for the extrapolation of sinusoids since the basis functions used in the expansion have a sinusoidal form. We have found that when the input signal $x[n]$ is a harmonic process, the extrapolated output $x_e[n]$ can be modelled (on the average) as

$$x_e[n] = x[n] \cdot w_A[n] \tag{4}$$

where $x[n]$ has the form

$$x[n] = \sum_{p=1}^{P} A_p \cdot e^{j(n\omega_p + \phi_p)} \tag{5}$$

with uncorrelated uniform random variables (amplitudes, frequencies and phases). Clearly, $w_A[n]$ is the desired AWNE imposed window [4].

Before performing any statistical analysis to estimate the imposed window, we need to first collect ensembles of input and output signals following the setup of figure 2, where it is understood that only L samples from $x[n]$ are used.

We have investigated three estimators used to obtain the AWNE imposed window statistically, they are:

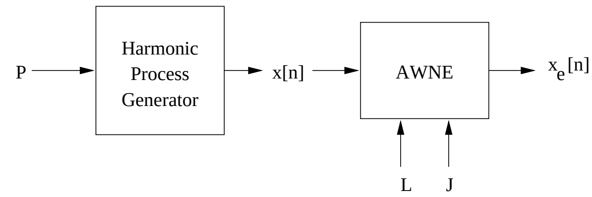

Figure 2: Signal collection process for statistical analysis.

the magnitude estimator which computes the average magnitude of the extrapolated outputs; the ratio estimator which obtains the estimated window through the average ratio between the extrapolated signal and the original harmonic signal; and the variance estimator which uses the autocorrelation of the extrapolated signal, the model defined in Eqn.(4), and the statistical properties of complex sinusoids to define the AWNE imposed window [5]. Among all three estimators, the ratio estimator produces the best estimate of the imposed window when the original signal is purely a sum of complex sinusoids [4], but fails when the signal is corrupted with noise [5]. The magnitude and the variance estimators are very similar since the latter uses the average squared magnitude of the extrapolated output. The variance estimator can produce the correct amplitude of the imposed window [4] and is the most tractable one to do analysis.

In general the variance estimator uses the autocorrelation of $x_e[n]$, $E\left\{(x[n_1] \cdot w[n_1])\,(x[n_2] \cdot w[n_2])^*\right\}$, to find an expression for the imposed window. Using conditional probability arguments and the fundamental expectation theorem, the variance of the output can be obtained from the autocorrelation to give [4]

$$E\left\{|x_e[n]|^2\right\} = \sigma_{x_e}^2[n] = w_A^2[n] \cdot \frac{P}{3} \tag{6}$$

and therefore, the imposed window estimate is

$$\hat{w}_A[n] = \sqrt{\frac{3}{P}} \cdot \sigma_{x_e}[n] \tag{7}$$

where P is the number of sinusoids present in the harmonic process and $\sigma_{x_e}[n]$ is the standard deviation of the extrapolation.

An existing window, the squared Tukey window [6], was found to closely match the estimate when the input signal is a sum of complex sinusoids and a Hamming window is used as $p[n]$ in AWNE. Figure 3 shows the imposed window estimate obtained using a sum of 3 complex sinusoids together with a squared Tukey window of the same size. This imposed window estimate is obtained from an ensemble of 500 realizations.

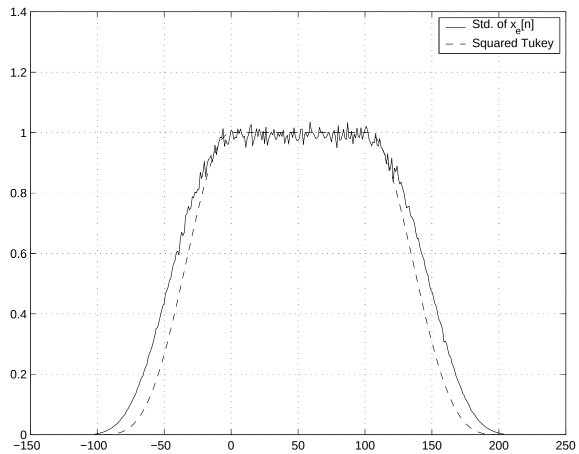

Figure 3: Standard deviation of $x_e[n]$ vs. squared Tukey window of length 307.

4. IMPOSED WINDOW ESTIMATION IN THE PRESENCE OF NOISE

From the previous section, we can see that $\hat{w}_A[n]$ depends on the sum of sinusoids input signal $x[n]$ and the window type used by the AWNE method (Hamming). On the other hand, if the input signal to AWNE method is pure Gaussian noise ($N[n]$) with zero mean and σ_N^2 power, and assuming that the model in Eqn.(4) holds, we find that the variance estimator is

$$E\{x_{Ne}[n_1] \cdot x_{Ne}^*[n_2]\} = \begin{cases} \sigma_N^2 \cdot w_N^2[n_1], & n_1 = n_2 \\ 0, & n_1 \neq n_2 \end{cases} \tag{8}$$

Following the same procedure as in Eqn.(6), the imposed window estimate, $\hat{w}_N[n]$, when Gaussian noise is extrapolated can be obtained from the extrapolated data as

$$\hat{w}_N[n] = \frac{1}{\sigma_N}\left[E\left\{|x_{Ne}[n]|^2\right\}\right]^{\frac{1}{2}} \tag{9}$$

The imposed window $\hat{w}_N[n]$ is data dependent as in the harmonic process case and its shape is dictated by the AWNE processing and the Gaussian noise input. In addition, the dependency of the imposed window on the input signal to be extrapolated and the window ($p[n]$) type in the AWNE algorithm makes the shape of the imposed window particular for every type of signal. Therefore, when there is additive Gaussian noise present in the input signal,

$$y[n] = x[n] + N[n] \tag{10}$$

the model defined in Eqn.(4) cannot be used since AWNE is not a linear process, consequently, the model in Eqn.(4) does not always represent the AWNE behavior. From the previous derivations, we also know that the imposed window produced by the AWNE algorithm is data dependent. We will see later that the windows associated with the harmonic process and the noise are different (i.e. $\hat{w}_A[n] \neq \hat{w}_N[n]$).

In our search to capture the AWNE behavior, a natural assumption could lead to the next model (where $w_x = w_A$),

$$x_e[n] = (w_x[n] + w_N[n]) \cdot (x[n] + N[n]) \tag{11}$$

However, the model in Eqn.(11) still assumes a linear treatment of the input and makes an assumption that contradicts the model in Eqn.(4). This is because under Eqn.(11) the imposed windows $w_x[n]$ and $w_N[n]$ have contributions from the signals $N[n]$ and $x[n]$, respectively, which is inconsistent with the data dependency assumption of the imposed window. In addition, the experimental procedure of extracting the imposed window using the variance from the extrapolated signals gives results that contradict the variance of Eqn.(11).

A model that is based on the assumptions of nonlinearity in the AWNE processing and data dependency of the imposed window is the following

$$x_e[n] = w_x[n] \cdot x[n] + w_N[n] \cdot N[n] \tag{12}$$

This model tries to capture the term dependency in the imposed window and does not assume linearity but rather an independent processing of the uncorrelated inputs in Eqn.(10). We will see that our results will be consistent with this assumption. By applying the variance estimator $E\left\{|x_e[n]|^2\right\}$ or more generally $E\{x_e[n_1] \cdot x_e^*[n_2]\}$, and using the assumed form of the extrapolated signal in Eqn.(12), we can see that the autocorrelation of the AWNE output is

$$\begin{aligned} &E\{(w_x[n_1] \cdot x[n_1] + w_N[n_1] \cdot N[n_1]) \\ &\quad \cdot (w_x^*[n_2] \cdot x^*[n_2] + w_N^*[n_2] \cdot N^*[n_2]) = \\ &\quad E\{(w_x[n_1] \cdot w_x^*[n_2] \cdot x[n_1] \cdot x^*[n_2])\} \\ &\quad + E\{(w_N[n_1] \cdot w_N^*[n_2] \cdot N[n_1] \cdot N^*[n_2])\} \end{aligned} \tag{13}$$

Since both the sum of complex sinusoids and the Gaussian noise are zero mean processes, and assuming real-valued windows, Eqn.(13) can be simplified using Eqns.(6) and (8) to give

$$E\{x_e[n_1] \cdot x_e^*[n_2]\} = \begin{cases} w_x^2[n_1] \cdot \frac{P}{3} + \sigma_N^2 \cdot w_N^2[n_1], & n_1 = n_2 \\ 0, & n_1 \neq n_2 \end{cases} \tag{14}$$

and therefore, the variance of the extrapolated signal becomes

$$\sigma_{x_e}^2[n] = w_x^2[n] \cdot \frac{P}{3} + \sigma_N^2 \cdot w_N^2[n] \tag{15}$$

Therefore, if we now define the imposed window as the standard deviation of the extrapolated output, it is given by

$$w_{x+N}[n] = \sqrt{w_x^2[n] \cdot \frac{P}{3} + \sigma_N^2 \cdot w_N^2[n]} \tag{16}$$

Thus, the new window has contributions from the two imposed windows defined in our assumed form of Eqn.(12). This suggests that we can obtain the form of the imposed window in Eqn.(16) by finding each contribution separately using the relationship between $w_{x+N}[n]$, $w_x^2[n]$ and $w_N^2[n]$. This tells us that if the variance of the Gaussian noise is small, the imposed window of the sum of complex sinusoids will dominate the form of the resulting imposed window.

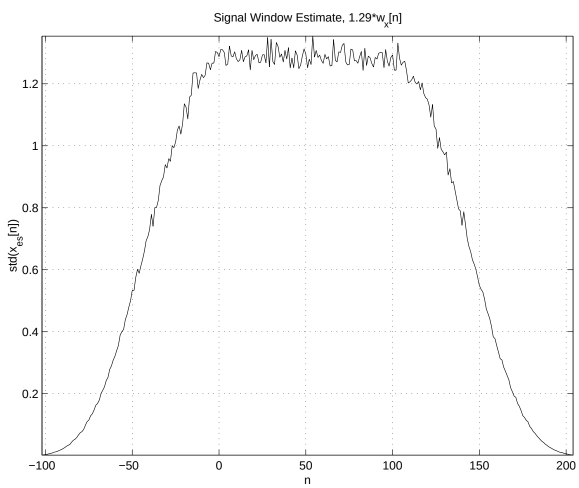

Figure 4: Imposed window estimate for harmonic process with 5 sinusoids.

Figures 4, 5, and 6 show the actual imposed window estimates of pure sinusoids, of pure noise, and of sinusoids plus noise respectively. First, 500 realizations of sums of complex sinusoids and of Gaussian noise are collected. The sum of sinusoids and the Gaussian noise are extrapolated both separately and together to produce ensembles of $x_{e_x}[n]$, $x_{eN}[n]$ and $x_{e_{x+N}}[n]$ for the statistical analysis. For this experiment, the parameter P is chosen to be 5, therefore, the signal window $w_x[n]$ in figure 4 is scaled by the constant $\sqrt{\frac{5}{3}}$ or 1.29. As mentioned before, the imposed window is input dependent, and therefore, figure 5 looks different than figure 4. Figure 6 shows the estimate of the imposed window

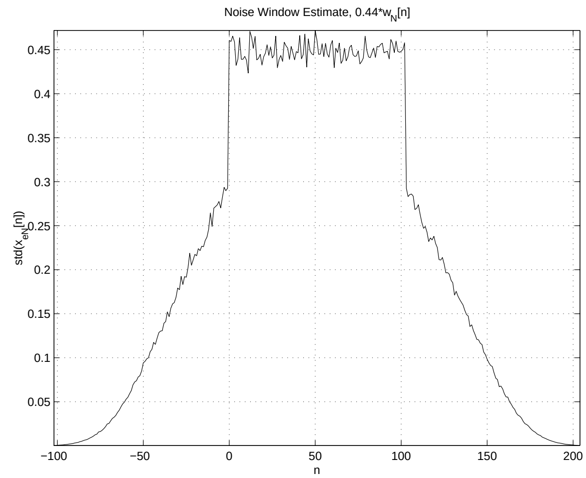

Figure 5: Window estimate for Gaussian noise with $\sigma_N^2 = 0.2$.

using the standard deviation of the extrapolated outputs for the sum of both terms as in Eqn.(12). On the other hand, figure 7 shows the window formed based on the relation stated in Eqn.(16).

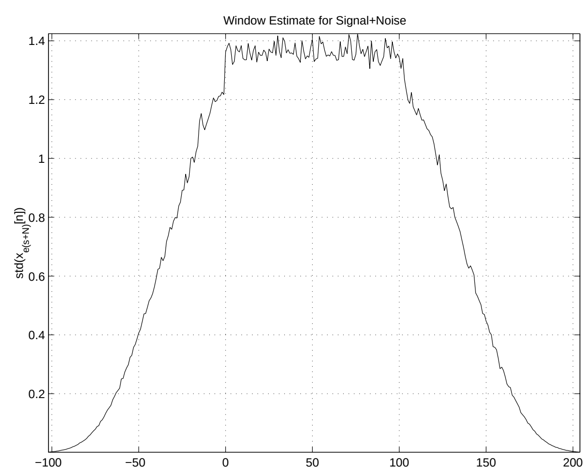

Figure 6: Window estimate for extrapolation of harmonic process ($P = 5$) plus Gaussian noise ($\sigma_N^2 = 0.2$).

Unlike the noiseless case, the AWNE imposed window here does not closely match an existing window type. However, if we consider the contribution from the noise window $w_N[n]$ to be small, we can ignore it and focus only on the signal window $w_x[n]$. Furthermore, for most actual signals, such as SAR data, we cannot separate the sinusoidal signals from the noise to analyze them independently. Therefore, for the case of sums of sinusoids in noise, the squared Tukey window can be used to model the overall AWNE imposed

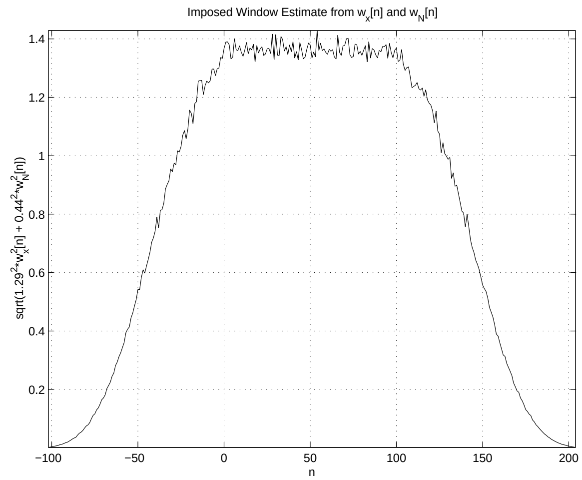

Figure 7: Window estimate formed by combining $w_x[n]$ and $w_N[n]$.

window.

5. VERIFICATION OF THE SQUARED TUKEY MODEL

In order to show that the squared Tukey window is a suitable replacement for the AWNE imposed window, the Normalized Inner Product (NIP) and the Normalized Mean-Squared-Error ($NMSE$) metrics are used [4]. These two metrics are defined as

$$NIP = \frac{\sum (x_e[n] \cdot x_w^*[n])}{\sqrt{\sum |x_e[n]|^2} \cdot \sqrt{\sum |x_w[n]|^2}} \tag{17}$$

and

$$NMSE = \frac{\sum |x_e[n] - x_w[n]|^2}{\sum |x_w[n]|^2} \tag{18}$$

where

$$x_w[n] = y[n] \cdot w_{Tukey^2}[n]$$
$$for \; -\frac{N_e - L}{2} \leq n \leq \frac{N_e + L}{2} - 1 \tag{19}$$

The NIP measures the coherent match between the two signals and the $NMSE$ calculates the % energy of their difference.

Table 1 shows the results of the evaluation using the two metrics under different noise variance situations. These results for each variance level are the averages for 1000 experiments. Notice that even under a high noise variance ($\sigma_N^2 = 2$) situation, the squared Tukey window can still provide a NIP of 80% match on the average. However, the $NMSE$ is up to about 38% of the energy of the original signal. Note that in Eqn.(19), we compare $x_e[n]$ with $y[n]$ which includes the additive noise. This comparison is consistent with the evaluation of AWNE applied to SAR phase history data described in the next section.

σ_N^2	0.125	0.250	0.5	1.0	2.0
$\lvert NIP_{T^2} \rvert$	0.926	0.902	0.868	0.834	0.798
MSE_{T^2}	0.149	0.193	0.254	0.315	0.378

Table 1: NIP and $NMSE$ verification of squared Tukey window model under different noise variance

6. SAR DATA VERIFICATION OF IMPOSED WINDOW

In order to obtain the imposed window estimate for SAR phase history data, 3880 profiles of spotlight SAR phase history records, each of length 99, are collected. From each profile, a subset of 33 samples is used to extrapolate to length 97. The variance estimator is then used on the 3880 extrapolated profiles to find the imposed window. Figure 8 shows both the actual imposed window estimate and the squared Tukey window. Notice that away from the middle portion of the estimate, the squared Tukey window matches the estimate closely. This gives us confidence that the squared Tukey window can be used to model this estimate. Furthermore, SAR phase history data seems to behave as a sum of sinusoids embedded in noise [7].

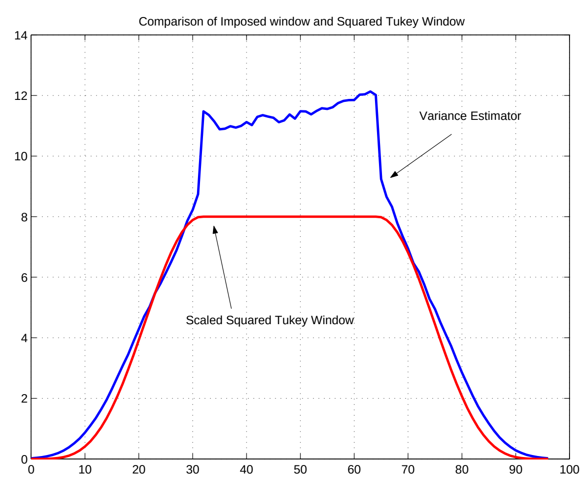

Figure 8: Comparison of the squared Tukey window and the imposed window estimate.

When applying the NIP and $NMSE$ metrics to evaluate the usage of the squared Tukey window as the overall imposed window for SAR phase history data, results show that there is an average of 76.12% match in coherent shape but the $NMSE$ is up to about 44%. Assuming that the squared Tukey window is a suitable model for the imposed window, we can use it to remove the imposed windowing effect from the extrapolated signal. Figure 9 shows the SAR spectrum before and after removing the squared Tukey effect from the extrapolated profile. Notice that after removing the windowing effect, the prominent peaks of the spectrum are sharper, thus the result has higher resolution.

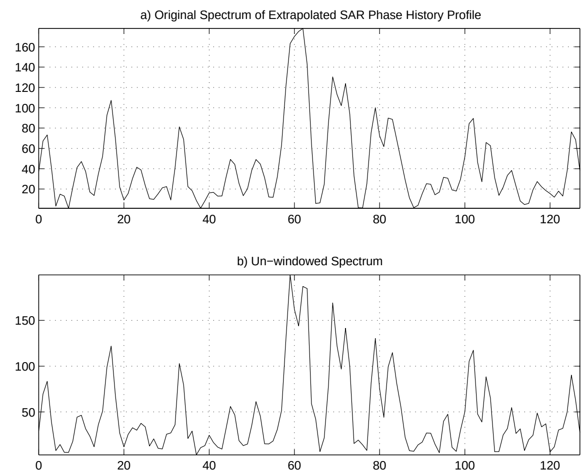

Figure 9: Extrapolated phase history spectrum vs. unwindowed spectrum.

7. CONCLUSIONS AND FUTURE WORK

We have introduced, estimated and modeled the window effects imposed by the AWNE extrapolation method for pure sum of sinusoids and for sum of sinusoids embedded in noise. We used a variance estimator on the extrapolated signals to obtain the window estimates. We also found that an existing parametric window, the squared Tukey window, models well the imposed window for low-noise cases. Two signal comparison metrics are used to show a very good match when the noise power is low. In addition, the metrics where applied to SAR profiles when the squared Tukey was assumed as the imposed window resulting in a 76.12% coherence shape match between the original and the extrapolated profiles and a 44% energy difference with respect to the original profiles. With the squared Tukey window, we performed un-windowing on SAR phase history profile data to demonstrate its applicability when using AWNE extrapolation to achieve superresolution.

Future work consists of doing a constructive analysis of the algorithm to confirm our experimental findings. We seek to improve its spectral detection capabilities in order to develope an estimation scheme that can be used in SAR image formation and automatic target recognition.

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